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Generalized Distance and Existence Theorems in Complete Metric Spaces

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In this paper, we first introduce the concept of τ-distance on a metric space, which is a generalized concept of both w-distance and Tataru's distance. We also improve the generalizations of the Banach contraction principle, Caristi's fixed point theorem, Ekeland's variational principle, and the nonconvex minimization theorem according to Takahashi. Further we discuss the relation between w-distance and Tataru's distance. © 2001 Academic Press

Key Words: fixed point; contractive mapping; Ekeland's variational principle; w-distance; Tataru's distance.

1. INTRODUCTION

The Banach contraction principle [1], Ekeland's variational principle [3], and Caristi's fixed point theorem [2] are forceful tools in nonlinear analysis, control theory, economic theory, and global analysis. These theorems are extended by several authors.

Let X be a metric space with metric d. Then a function p from $X \times X$ into $\mathbb{R}_+$ is called a w-distance on X if it satisfies the following:

(w1) $p(x, z) \le p(x, y) + p(y, z)$ for all $x, y, z \in X$;

(w2) p is lower semicontinuous in its second variable;

(w3) for each $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \le \delta$ and $p(z, y) \le \delta$ imply $d(x, y) \le \varepsilon$.

The metric d is a w-distance on X. The concept of w-distance was first introduced by Kada *et al.* [5]. They gave some examples of w-distance and

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improved Caristi's fixed point theorem, Ekeland's variational principle, and the nonconvex minimization theorem according to Takahashi [12]. Also the fixed point theorem for contractive mappings with respect to a w-distance was proved in [11]. See also [8–10].

Let X be a subset of a Banach space and let $\{T(t): t \in \mathbb{R}_+\}$ be a strongly continuous semigroup of nonexpansive mappings on X, i.e.,

(sg1) For each $t \in \mathbb{R}_+$, $T(t)$ is a nonexpansive mapping on X;

(sg2) $T(0)x = x$ for all $x \in X$;

(sg3) $T(s+t) = T(s) \circ T(t)$ for all $s, t \in \mathbb{R}_+$;

(sg4) for each $x \in X$, the mapping $T(\cdot)$ from $\mathbb{R}_+$ into X is continuous.

In [14], Tataru introduced the distance

$$p(x,y) = \inf\{t + \|T(t)x - y\| : t \in \mathbb{R}_+\}$$

for all $x, y \in X$. Further he proved another generalization of Ekeland's variational principle. Using it, he studied Hamilton–Jacobi equations. In [7], Kocan and Święch used Tataru's results to study optimization. Note that Tataru's distance is not necessarily a w-distance.

Let X be a metric space with metric d, let h be a nondecreasing function from $\mathbb{R}_+$ into itself such that $\int_0^\infty (1/(1+h(r)))\,dr = \infty$, and let $z_0 \in X$ be fixed. Zhong [16] considered the following function p from $X \times X$ into $\mathbb{R}_+$ defined by

$$p(x,y) = \frac{d(x,y)}{1 + h(d(z_0,x))}$$

for all $x, y \in X$ and proved the other generalization of Ekeland's variational principle.

In this paper, we first introduce the concept of τ-distance on a metric space, which is a generalized concept of both w-distance and Tataru's distance. We also improve the generalizations of the Banach contraction principle, Caristi's fixed point theorem, Ekeland's variational principle, and the nonconvex minimization theorem according to Takahashi. Further we discuss the relation between w-distance and Tataru's distance.

2. τ-DISTANCE

Throughout this paper we denote by $\mathbb{N}$ and $\mathbb{R}_+$ the set of positive integers and nonnegative real numbers, respectively. We first give a definition. Let X be a metric space with metric d. Then a function p from

$X \times X$ into $\mathbb{R}_+$ is called τ-distance on X if there exists a function η from $X \times \mathbb{R}_+$ into $\mathbb{R}_+$ and the following are satisfied:

($\tau 1$) $p(x,z) \le p(x,y) + p(y,z)$ for all $x, y, z \in X$;

($\tau 2$) $\eta(x,0) = 0$ and $\eta(x,t) \ge t$ for all $x \in X$ and $t \in \mathbb{R}_+$, and η is concave and continuous in its second variable;

($\tau 3$) $\lim_n x_n = x$ and $\lim_n \sup\{\eta(z_n, p(z_n, x_m)) : m \ge n\} = 0$ imply $p(w,x) \le \liminf_n p(w, x_n)$ for all $w \in X$;

($\tau 4$) $\lim_n \sup\{p(x_n, y_m) : m \ge n\} = 0$ and $\lim_n \eta(x_n, t_n) = 0$ imply $\lim_n \eta(y_n, t_n) = 0$;

($\tau 5$) $\lim_n \eta(z_n, p(z_n, x_n)) = 0$ and $\lim_n \eta(z_n, p(z_n, y_n)) = 0$ imply $\lim_n d(x_n, y_n) = 0$.

Remark. We may replace ($\tau 2$) by the following ($\tau 2$)$'$

($\tau 2$)$'$ $\inf\{\eta(x,t) : t > 0\} = 0$ for all $x \in X$, and η is nondecreasing in its second variable.

Proof. Assume θ is a function from $X \times \mathbb{R}_+$ into $\mathbb{R}_+$ satisfying $(\tau 2)'_\theta$ and $(\tau 3)_\theta - (\tau 5)_\theta$. We define a function η from $X \times \mathbb{R}_+$ into $\mathbb{R}_+$ by

$$\eta(x,t) = t + \sup\left\{ \sum_{i=1}^{n} \alpha_i \cdot \min\{\theta(x,s_i), 1\} : t = \sum_{i=1}^{n} \alpha_i s_i,\ s_i \ge 0, \right.$$
$$\left. \alpha_i > 0,\ \sum_{i=1}^{n} \alpha_i = 1 \right\}$$

for all $x \in X$ and $t \in \mathbb{R}_+$. We shall show such η satisfies ($\tau 2$)–($\tau 5$). Since $\eta(x,t) \ge \min\{\theta(x,t), 1\}$ for all $x \in X$ and $t \in \mathbb{R}_+$, ($\tau 3$) and ($\tau 5$) hold. It is clear that $\eta(x,0) = 0$ and $\eta(x,t) \ge t$ for all $x \in X$, and η is concave in its second variable. So, we shall prove $\eta(x, \cdot)$ is continuous at 0 for all $x \in X$. Assume that there exist $\varepsilon_1 > 0$, $x \in X$, and a sequence $\{t_n\}$ of $\mathbb{R}_+$ such that $\eta(x, t_n) > 2\varepsilon_1$ for all $n \in \mathbb{N}$ and $\lim_n t_n = 0$. Then there exists $\delta > 0$ such that $\theta(x, \delta) \le \varepsilon_1$. Fix $n_1 \in \mathbb{N}$ with $t_{n_1} + t_{n_1}/\delta \le \varepsilon_1$. Since $\eta(x, t_{n_1}) > 2\varepsilon_1$, there exist $\alpha_1, \alpha_2, \dots, \alpha_k > 0$ and $s_1, s_2, \dots, s_k \ge 0$ such that $t_{n_1} = \sum_{i=1}^k \alpha_i s_i$, $\sum_{i=1}^k \alpha_i = 1$ and $t_{n_1} + \sum_{i=1}^k \alpha_i \cdot \min\{\theta(x, s_i), 1\} > 2\varepsilon_1$. Since $\delta \cdot \sum\{\alpha_i : s_i \ge \delta\} \le t_{n_1}$, we obtain

$$\begin{aligned} 2\varepsilon_1 &< t_{n_1} + \sum_{i=1}^{k} \alpha_i \cdot \min\{\theta(x,s_i), 1\} \\ &\le t_{n_1} + \sum_{s_i < \delta} \alpha_i \theta(x, s_i) + \sum_{s_i \ge \delta} \alpha_i \\ &\le t_{n_1} + \sum_{s_i < \delta} \alpha_i \varepsilon_1 + \sum_{s_i \ge \delta} \alpha_i \\ &\le t_{n_1} + \varepsilon_1 + t_{n_1}/\delta \le 2\varepsilon_1. \end{aligned}$$

This is a contradiction. Let us prove $(\tau 4)$. Assume that $\lim_n \sup\{p(x_n, y_m) : m \ge n\} = 0$, $\lim_n \eta(x_n, t_n) = 0$, and $\limsup_n \eta(y_n, t_n) > 0$. Put $\varepsilon_2 = \min\{1, \limsup_n \eta(y_n, t_n)/3\}$. From $(1 - \varepsilon_2/2)(1/(2 - \varepsilon_2)) + (\varepsilon_2/2)(1/\varepsilon_2) = 1$, we obtain

$$\begin{aligned} t_n &+ \left(1 - \frac{\varepsilon_2}{2}\right) \cdot \min\left\{\theta\left(x_n, \frac{t_n}{2 - \varepsilon_2}\right), 1\right\} + \frac{\varepsilon_2}{2} \cdot \min\left\{\theta\left(x_n, \frac{t_n}{\varepsilon_2}\right), 1\right\} \\ &\le \eta(x_n, t_n) \end{aligned}$$

and hence $\lim_n t_n = 0$ and $\lim_n \theta(x_n, t_n/\varepsilon_2) = 0$. From $(\tau 4)_\theta$, we have $\lim_n \theta(y_n, t_n/\varepsilon_2) = 0$. Fix $n_2 \in \mathbb{N}$ with $t_{n_2} + \theta(y_{n_2}, t_{n_2}/\varepsilon_2) \le \varepsilon_2$ and $\eta(y_{n_2}, t_{n_2}) > 2\varepsilon_2$. Then there exist $\alpha_1, \alpha_2, \dots, \alpha_l > 0$ and $s_1, s_2, \dots, s_l \ge 0$ such that $t_{n_2} = \sum_{i=1}^l \alpha_i s_i$, $\sum_{i=1}^l \alpha_i = 1$, and $t_{n_2} + \sum_{i=1}^l \alpha_i \cdot \min\{\theta(y_{n_2}, s_i), 1\} > 2\varepsilon_2$. Since $\sum\{\alpha_i : s_i \ge t_{n_2}/\varepsilon_2\} \le \varepsilon_2$, we obtain

$$\begin{aligned} 2\varepsilon_2 &< t_{n_2} + \sum_{i=1}^{l} \alpha_i \cdot \min\{\theta(y_{n_2}, s_i), 1\} \\ &\le t_{n_2} + \sum_{s_i < t_{n_2}/\varepsilon_2} \alpha_i \cdot \theta(y_{n_2}, t_{n_2}/\varepsilon_2) + \sum_{s_i \ge t_{n_2}/\varepsilon_2} \alpha_i \\ &\le t_{n_2} + \theta(y_{n_2}, t_{n_2}/\varepsilon_2) + \varepsilon_2 \le 2\varepsilon_2. \end{aligned}$$

This is a contradiction. ∎

We next prove two propositions, which show that the concept of τ-distance is a generalized concept of both w-distance and Tataru's distance.

PROPOSITION 1. *Let p be a w-distance on a metric space X. Then p is also a τ-distance on X.*

Proof. We define a function η from $X \times \mathbb{R}_+$ into $\mathbb{R}_+$ by $\eta(x, t) = t$ for all $x \in X$ and $t \in \mathbb{R}_+$. Then $(\tau 2)$ and $(\tau 4)$ hold. Clearly $(w2)$ implies $(\tau 3)$. Further $(w1)$ and $(w3)$ are equivalent to $(\tau 1)$ and $(\tau 5)$, respectively. Therefore p is a τ-distance on X. ∎

PROPOSITION 2. *Let $\{T(t) : t \in \mathbb{R}_+\}$ be a strongly continuous semigroup of nonexpansive mappings on a subset X of a Banach space. Then Tataru's distance p on X is also a τ-distance on X.*

Proof. We first prove $(\tau 1)$. From

$$\begin{aligned} p(x, z) &\le s + t + \|T(s + t)x - z\| \\ &\le s + t + \|T(s + t)x - T(t)y\| + \|T(t)y - z\| \\ &\le s + t + \|T(s)x - y\| + \|T(t)y - z\| \end{aligned}$$

for all $s, t \in \mathbb{R}_+$, we obtain $p(x,z) \le p(x,y) + p(y,z)$. We define a function η from $X \times \mathbb{R}_+$ into $\mathbb{R}_+$ by

$$\eta(x,t) = t + \max\{\|T(s)x - x\| : 0 \le s \le t\}$$

for all $x \in X$ and $t \in \mathbb{R}_+$. Then $(\tau 2)'$ holds. Suppose $\{x_n\}$ and $\{y_n\}$ are sequences of X which converge to x and y in X, respectively. Fix $\alpha_1 > p(x,y)$. Then there exists $t_1 \in \mathbb{R}_+$ such that $t_1 + \|T(t_1)x - y\| < \alpha_1$. For sufficient large $n \in \mathbb{N}$, we obtain

$$\begin{aligned} p(x_n, y_n) &\le t_1 + \|T(t_1)x_n - y_n\| \\ &\le t_1 + \|T(t_1)x - y\| + \|T(t_1)x - T(t_1)x_n\| + \|y - y_n\| \\ &\le t_1 + \|T(t_1)x - y\| + \|x - x_n\| + \|y - y_n\| < \alpha_1 \end{aligned}$$

and hence $p(x,y) \ge \limsup_n p(x_n, y_n)$. Fix $\alpha_2 > \liminf_n p(x_n, y_n)$. Then there exist $k_1 \in \mathbb{N}$ and $t_2 \in \mathbb{R}_+$ such that

$$t_2 + \|T(t_2)x_{k_1} - y_{k_1}\| + \|x - x_{k_1}\| + \|y - y_{k_1}\| < \alpha_2.$$

So we obtain

$$\begin{aligned} p(x,y) &\le t_2 + \|T(t_2)x - y\| \\ &\le t_2 + \|T(t_2)x_{k_1} - y_{k_1}\| + \|T(t_2)x - T(t_2)x_{k_1}\| + \|y - y_{k_1}\| \\ &\le t_2 + \|T(t_2)x_{k_1} - y_{k_1}\| + \|x - x_{k_1}\| + \|y - y_{k_1}\| < \alpha_2 \end{aligned}$$

and hence $p(x,y) \le \liminf_n p(x_n, y_n)$. Therefore p is continuous on $X \times X$. This implies $(\tau 3)$. Let us prove $(\tau 4)$. Fix $x, y \in X$, $t \in \mathbb{R}_+$, and $\alpha_3 > p(x,y)$. Then there exists $t_3 \in \mathbb{R}_+$ such that $t_3 + \|T(t_3)x - y\| \le \alpha_3$. For $s \in [0,t]$, we have

$$\begin{aligned} & t + \|T(s)y - y\| \\ &\quad \le t + \|T(s)y - T(s+t_3)x\| + \|T(s+t_3)x - T(t_3)x\| \\ &\qquad + \|T(t_3)x - y\| \\ &\quad \le 2\|T(t_3)x - y\| + t + \|T(s)x - x\| \le 2\alpha_3 + \eta(x,t) \end{aligned}$$

and hence $\eta(y,t) \le 2p(x,y) + \eta(x,t)$. This implies $(\tau 4)$. Finally we show $(\tau 5)$. Suppose $\lim_n \eta(z_n, p(z_n, x_n)) = 0$ and $\lim_n \eta(z_n, p(z_n, y_n)) = 0$. Then since η is continuous in its second variable, there exists a sequence $\{t_n\}$ of

$\mathbb{R}_+$ such that $\eta(z_n, t_n + \|T(t_n)z_n - x_n\|) \le \eta(z_n, p(z_n, x_n)) + 1/n$. So we have

$$\begin{aligned}\|x_n - z_n\| &\le \|T(t_n)z_n - x_n\| + \|T(t_n)z_n - z_n\| \\ &\le t_n + \|T(t_n)z_n - x_n\| \\ &\quad + \max\{\|T(s)z_n - z_n\| : 0 \le s \le t_n + \|T(t_n)z_n - x_n\|\} \\ &= \eta(z_n, t_n + \|T(t_n)z_n - x_n\|) \\ &\le \eta(z_n, p(z_n, x_n)) + 1/n\end{aligned}$$

and hence $\lim_n \|x_n - z_n\| = 0$. Similarly we obtain $\lim_n \|y_n - z_n\| = 0$. Therefore $\lim_n \|x_n - y_n\| = 0$. ∎

Remark. For $s \in [0, t]$,

$$\begin{aligned}\|T(s)x - x\| &\le \|T(s)x - T(s)y\| + \|T(s)y - y\| + \|y - x\| \\ &\le 2\|x - y\| + \|T(s)y - y\|.\end{aligned}$$

So we obtain $\eta(x, t) \le 2\|x - y\| + \eta(y, t)$ and hence $|\eta(x, t) - \eta(y, t)| \le 2\|x - y\|$ for all $x, y \in X$ and $t \in \mathbb{R}_+$. Therefore η is continuous in its first variable. The following proposition says that Tataru's distance is a w-distance if X is compact.

PROPOSITION 3. *Let X be a compact metric space, let p be a τ-distance on X, and let η be a function from $X \times \mathbb{R}_+$ into $\mathbb{R}_+$ satisfying $(\tau 2)'$, $(\tau 4)$, and $(\tau 5)$. Suppose p is lower semicontinuous in its second variable and η is continuous in its first variable. Then p is a w-distance on X.*

Proof. Clearly $(w1)$ and $(w2)$ hold. Let us prove $(w3)$. Assume $(w3)$ does not hold. Then there exist $\varepsilon > 0$ and sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ of X such that $p(z_n, x_n) \le p(z_n, y_n) \le 1/n$ and $d(x_n, y_n) \ge \varepsilon$ for all $n \in \mathbb{N}$. From $(\tau 5)$ there exists $\delta > 0$ such that $\eta(z_n, p(z_n, y_n)) \ge \delta$ for all $n \in \mathbb{N}$. Since X is compact, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ which converges to some point z in X. For $k, m \in \mathbb{N}$ with $n_k > m$, we obtain $\eta(z_{n_k}, 1/m) \ge \eta(z_{n_k}, 1/n_k) \ge \eta(z_{n_k}, p(z_{n_k}, y_{n_k})) \ge \delta$. Hence we have $\eta(z, 1/m) \ge \delta$ for all $m \in \mathbb{N}$. This contradicts $(\tau 2)'$. ∎

The following proposition is connected with Zhong's result [16].

PROPOSITION 4. *Let X be a metric space with metric d, let p be a w-distance on X, let h be a nondecreasing function from $\mathbb{R}_+$ into itself such that $\int_0^\infty (1/(1 + h(r)))\, dr = \infty$, and let $z_0 \in X$ be fixed. Then a function q*

from $X \times X$ into $\mathbb{R}_+$ defined by

$$q(x,y) = \int_{p(z_0,x)}^{p(z_0,x)+p(x,y)} \frac{dr}{1+h(r)}$$

for all $x, y \in X$ is a τ-distance on X.

Proof. From $(w1)$, we have

$$\begin{aligned} q(x,z) &= \int_{p(z_0,x)}^{p(z_0,x)+p(x,z)} \frac{dr}{1+h(r)} \\ &\le \int_{p(z_0,x)}^{p(z_0,x)+p(x,y)+p(y,z)} \frac{dr}{1+h(r)} \\ &= \int_{p(z_0,x)}^{p(z_0,x)+p(x,y)} \frac{dr}{1+h(r)} + \int_{p(z_0,x)+p(x,y)}^{p(z_0,x)+p(x,y)+p(y,z)} \frac{dr}{1+h(r)} \\ &\le \int_{p(z_0,x)}^{p(z_0,x)+p(x,y)} \frac{dr}{1+h(r)} + \int_{p(z_0,y)}^{p(z_0,y)+p(y,z)} \frac{dr}{1+h(r)} \\ &= q(x,y) + q(y,z) \end{aligned}$$

for all $x, y, z \in X$ and hence $(\tau 1)_q$ holds. We define a function η from $X \times \mathbb{R}_+$ into $\mathbb{R}_+$ by

$$\eta(x,t) = t + h\big(p(z_0,x) + 1\big) \cdot t$$

for all $x \in X$ and $t \in \mathbb{R}_+$. Then $(\tau 2)$ holds. From $(w2)$, q is lower semicontinuous in its second variable. So, $(\tau 3)_{q,\eta}$ holds. We next prove $(\tau 4)_{q,\eta}$. Assume $\lim_n \sup\{q(x_n, y_m) : m \ge n\} = 0$ and $\lim_n \eta(x_n, t_n) = 0$. Then there exists $n_1 \in \mathbb{N}$ such that $\sup\{q(x_{n_1}, y_m) : m \ge n_1\} < 1$. Since $\int_0^\infty (1/(1+h(r)))\, dr = \infty$, we obtain $\{p(x_{n_1}, y_m)\}$ is bounded and hence $p(z_0, y_m) + 1 \le M$ for some $M > 0$. $\lim_n \eta(x_n, t_n) = 0$ implies $\lim_n t_n = 0$. So we have

$$\limsup_{n\to\infty} \eta(y_n, t_n) \le \lim_{n\to\infty} t_n + h(M) \cdot t_n = 0.$$

Let us prove $(\tau 5)_{q,\eta}$. Assume that $\lim_n \eta(z_n, q(z_n, x_n)) = 0$ and $\lim_n \eta(z_n, q(z_n, y_n)) = 0$. Without loss of generality, we may assume

$\eta(z_n, q(z_n, x_n)) \leq 1$ for all $n \in \mathbb{N}$. Then we obtain

$$\begin{aligned}\int_{p(z_0, z_n)}^{p(z_0, z_n)+p(z_n, x_n)} \frac{dr}{1+h(r)} &= q(z_n, x_n) \\ &= \frac{\eta(z_n, q(z_n, x_n))}{1+h(p(z_0, z_n)+1)} \\ &= \int_{p(z_0, z_n)}^{p(z_0, z_n)+\eta(z_n, q(z_n, x_n))} \frac{dr}{1+h(p(z_0, z_n)+1)} \\ &\leq \int_{p(z_0, z_n)}^{p(z_0, z_n)+\eta(z_n, q(z_n, x_n))} \frac{dr}{1+h(r)}\end{aligned}$$

and hence

$$\limsup_{n\to\infty} p(z_n, x_n) \leq \lim_{n\to\infty} \eta(z_n, q(z_n, x_n)) = 0.$$

Similarly we obtain $\lim_n p(z_n, y_n) = 0$. So we have $\lim_n d(x_n, y_n) = 0$ from $(w3)$. This completes the proof. ∎

The following proposition is connected with the results of Ume [15], Takahashi [13], and Kim *et al*. [6].

PROPOSITION 5. *Let X be a metric space with metric d and let p be a τ-distance on X. Let T be a mapping from X into itself satisfying that $\lim_n x_n = y$ and $\lim_n Tx_n = y$ imply $Ty = y$. Then a function q from $X \times X$ into $\mathbb{R}_+$ defined by*

$$q(x, y) = \max\{p(Tx, Ty), p(Tx, y)\}$$

for all $x, y \in X$ is also a τ-distance.

Proof. Let η be a function from $X \times \mathbb{R}_+$ into $\mathbb{R}_+$ satisfying $(\tau 2)$–$(\tau 5)$.

$$\begin{aligned}q(x, z) &= \max\{p(Tx, Tz), p(Tx, z)\} \\ &\leq \max\{p(Tx, Ty) + p(Ty, Tz), p(Tx, Ty) + p(Ty, z)\} \\ &\leq q(x, y) + q(y, z)\end{aligned}$$

for all $x, y, z \in X$ and hence $(\tau 1)_q$ holds. We define a function θ from $X \times \mathbb{R}_+$ into $\mathbb{R}_+$ by $\theta(x, t) = \eta(Tx, t)$ for all $x \in X$ and $t \in \mathbb{R}_+$. Then $(\tau 2)_\theta$ holds. Assume that $\lim_n x_n = x$ and $\lim_n \sup\{\theta(z_n, q(z_n, x_m)) : m \geq n\} = 0$. Then since $\lim_n \sup\{\eta(Tz_n, \max\{p(Tz_n, Tx_m), p(Tz_n, x_m)\}) : m \geq n\} = 0$, we have $\lim_n Tx_n = \lim_n x_n = x$ from $(\tau 5)$. By assumption, we obtain $Tx = x$. Therefore

$$q(w, x) = p(Tw, x) \leq \liminf_{n\to\infty} p(Tw, x_n) \leq \liminf_{n\to\infty} q(w, x_n)$$

for all $w \in X$. This implies $(\tau 3)_{q,\theta}$. Assume $\lim_n \sup\{q(x_n, y_m) : m \geq n\} = 0$ and $\lim_n \theta(x_n, t_n) = 0$. Since $\lim_n \sup\{p(Tx_n, Ty_m) : m \geq n\} = 0$ and $\lim_n \eta(Tx_n, t_n) = 0$, we obtain

$$\lim_{n\to\infty} \theta(y_n, t_n) = \lim_{n\to\infty} \eta(Ty_n, t_n) = 0.$$

This shows $(\tau 4)_{q,\theta}$. Let us prove $(\tau 5)_{q,\theta}$. Assume that $\lim_n \theta(z_n, q(z_n, x_n)) = 0$ and $\lim_n \theta(z_n, q(z_n, y_n)) = 0$. Then since $\lim_n \eta(Tz_n, p(Tz_n, x_n)) = 0$ and $\lim_n \eta(Tz_n, p(Tz_n, y_n)) = 0$, we obtain $\lim_n d(x_n, y_n) = 0$. ∎

The following proposition is connected with Jachymski's result [4].

PROPOSITION 6. *Let X be a metric space with metric d and let p be a τ-distance on X. Let φ be a nondecreasing and left continuous function from $\mathbb{R}_+$ into itself such that $0 < \varphi(s+t) \leq \varphi(s) + \varphi(t)$ for all $s \geq 0$ and $t > 0$. Then a function q from $X \times X$ into $\mathbb{R}_+$ defined by $q(x, y) = \varphi(p(x, y))$ for all $x, y \in X$ is also a τ-distance.*

Proof. Let η be a function from $X \times \mathbb{R}_+$ into $\mathbb{R}_+$ satisfying $(\tau 2)$–$(\tau 5)$. We have

$$\begin{aligned} q(x,z) &= \varphi(p(x,z)) \leq \varphi(p(x,y) + p(y,z)) \\ &\leq \varphi(p(x,y)) + \varphi(p(y,z)) = q(x,y) + q(y,z) \end{aligned}$$

for all x, y, z and hence $(\tau 1)_q$ holds. Define a nondecreasing function ξ from $\mathbb{R}_+$ into itself by

$$\xi(s) = \begin{cases} 0, & \text{if } s < \varphi(0), \\ \min\{\sup\{t : \varphi(t) \leq s\}, 1\}, & \text{if } s \geq \varphi(0) \end{cases}$$

for all $s \in \mathbb{R}_+$ and define a function θ from $X \times \mathbb{R}_+$ into $\mathbb{R}_+$ by $\theta(x, t) = \eta(x, \xi(t))$ for all $x \in X$ and $t \in \mathbb{R}_+$. Clearly $(\tau 2)'_\theta$ holds. Since

$$\theta(x, \varphi(t)) = \eta(x, \xi(\varphi(t))) \geq \eta(x, \min\{t, 1\}) \geq \min\{t, 1\},$$

we obtain $\theta(x, \varphi(t)) \geq \eta(x, t)$ if $\theta(x, \varphi(t)) < 1$. We next prove $(\tau 3)_{q,\theta}$. Assume that $\lim_n x_n = x$ and $\lim_n \sup\{\theta(z_n, q(z_n, x_m)) : m \geq n\} = 0$. Then since

$$\limsup_{n\to\infty} \sup_{m\geq n} \eta(z_n, p(z_n, x_m)) \leq \lim_{n\to\infty} \sup_{m \geq n} \theta(z_n, q(z_n, x_m)) = 0,$$

we get $p(w, x) \leq \liminf_n p(w, x_n)$ for all $w \in X$. Hence we have

$$\begin{aligned} q(w,x) &= \varphi(p(w,x)) \leq \varphi\Big(\liminf_{n\to\infty} p(w, x_n)\Big) \\ &\leq \liminf_{n\to\infty} \varphi(p(w, x_n)) = \liminf_{n\to\infty} q(w, x_n) \end{aligned}$$

for all $w \in X$. We next prove $(\tau 4)_{q,\theta}$. Assume that $\lim_n \sup\{q(x_n, y_m) : m \geq n\} = 0$ and $\lim_n \theta(x_n, t_n) = 0$. Then $\lim_n \sup\{p(x_n, y_m) : m \geq n\} = 0$ and $\lim_n \eta(x_n, \xi(t_n)) = 0$. So, we obtain $\lim_n \theta(y_n, t_n) = \lim_n \eta(y_n, \xi(t_n)) = 0$. Let us prove $(\tau 5)_{q,\theta}$. Assume $\lim_n \theta(z_n, q(z_n, x_n)) = 0$ and $\lim_n \theta(z_n, q(z_n, y_n)) = 0$. Then we have $\lim_n \eta(z_n, p(z_n, x_n)) = 0$ and $\lim_n \eta(z_n, p(z_n, y_n)) = 0$. Hence $\lim_n d(x_n, y_n) = 0$. This completes the proof. ∎

As a direct consequence of Proposition 6, we obtain the following, which is used in Section 5.

PROPOSITION 7. *Let p be a τ-distance on a metric space X and let c be a positive real number. Then a function q from $X \times X$ into $\mathbb{R}_+$ defined by $q(x, y) = c \cdot p(x, y)$ for all $x, y \in X$ is also a τ-distance on X.*

3. PROPERTIES OF τ-DISTANCE

In this section, we discuss some properties of τ-distance. We first give a definition. Let X be a metric space with metric d and let p be a τ-distance on X. Then a sequence $\{x_n\}$ of X is called p-Cauchy if there exist a function η from $X \times \mathbb{R}_+$ into $\mathbb{R}_+$ satisfying $(\tau 2)$–$(\tau 5)$ and a sequence $\{z_n\}$ of X such that $\lim_n \sup\{\eta(z_n, p(z_n, x_m)) : m \geq n\} = 0$. The following lemmas are crucial in the proofs of the theorems in Section 4 and Section 5.

LEMMA 1. *Let X be a metric space with metric d and let p be a τ-distance on X. If $\{x_n\}$ is a p-Cauchy sequence, then $\{x_n\}$ is a Cauchy sequence. Moreover, if $\{y_n\}$ is a sequence satisfying $\lim_n \sup\{p(x_n, y_m) : m \geq n\} = 0$, then $\{y_n\}$ is also a p-Cauchy sequence and $\lim_n d(x_n, y_n) = 0$.*

Proof. By assumption, there exist a function η from $X \times \mathbb{R}_+$ into $\mathbb{R}_+$ satisfying $(\tau 2)$–$(\tau 5)$ and a sequence $\{z_n\}$ of X such that $\lim_n \sup\{\eta(z_n, p(z_n, x_m)) : m \geq n\} = 0$. Then from $(\tau 5)$, we have $\lim_n \sup\{d(x_i, x_j) : j > i \geq n\} = 0$. This means $\{x_n\}$ is a Cauchy sequence. Moreover if $\{y_n\}$ is a sequence satisfying $\lim_n \sup\{p(x_n, y_m) : m \geq n\} = 0$, then we put $\alpha_n = \sup\{p(z_n, x_m) : m \geq n\}$ and $\beta_n = \sup\{p(x_i, y_j) : j \geq i \geq n\}$. Note that $\{\beta_n\}$ is a nonincreasing sequence and converges to 0. From $(\tau 2)$, there exist sequences $\{\delta_n\}$ and $\{\varepsilon_n\}$ of positive real numbers which converge to 0 and satisfy $\beta_1 \leq \delta_1$ and $\eta(z_n, \alpha_n + \delta_n) \leq \eta(z_n, \alpha_n) + \varepsilon_n$ for all $n \in \mathbb{N}$. Then we can define a mapping f from $\mathbb{N}$ into itself such that $f(n) \le n$ and $\beta_n \le \delta_{f(n)}$ for all $n \in \mathbb{N}$ and $\lim_n f(n) = \infty$. Now we obtain

$$\begin{aligned}
&\limsup_{n\to\infty} \sup_{m \ge n} \eta\big(z_{f(n)}, p(z_{f(n)}, y_m)\big) \\
&\quad \le \limsup_{n\to\infty} \sup_{m \ge n} \eta\big(z_{f(n)}, p(z_{f(n)}, x_n) + p(x_n, y_m)\big) \\
&\quad \le \limsup_{n\to\infty} \eta\big(z_{f(n)}, \alpha_{f(n)} + \beta_n\big) \\
&\quad \le \limsup_{n\to\infty} \eta\big(z_{f(n)}, \alpha_{f(n)} + \delta_{f(n)}\big) \\
&\quad \le \lim_{n\to\infty} \eta(z_{f(n)}, \alpha_{f(n)}) + \varepsilon_{f(n)} = 0.
\end{aligned}$$

Therefore $\{y_n\}$ is a p-Cauchy sequence. Since

$$\limsup_{n\to\infty} \eta\big(z_{f(n)}, p(z_{f(n)}, x_n)\big) \le \lim_{n\to\infty} \eta(z_{f(n)}, \alpha_{f(n)}) = 0,$$

we have $\lim_n d(x_n, y_n) = 0$. ∎

LEMMA 2. *Let X be a metric space with metric d and let p be a τ-distance on X. If a sequence $\{x_n\}$ of X satisfies $\lim_n p(z, x_n) = 0$ for some $z \in X$, then $\{x_n\}$ is a p-Cauchy sequence. Moreover, if a sequence $\{y_n\}$ of X also satisfies $\lim_n p(z, y_n) = 0$, then $\lim_n d(x_n, y_n) = 0$. In particular for $x, y, z \in X$, $p(z, x) = 0$ and $p(z, y) = 0$ imply $x = y$.*

Proof. Let η be a function from $X \times \mathbb{R}_+$ into $\mathbb{R}_+$ satisfying (τ2)–(τ5). From (τ2), note that $\lim_n p(z, x_n) = 0$ is equivalent to $\lim_n \eta(z, p(z, x_n)) = 0$. This shows that $\{x_n\}$ is a p-Cauchy sequence. Moreover if a sequence $\{y_n\}$ of X also satisfies $\lim_n p(z, y_n) = 0$, then $\lim_n \eta(z, p(z, y_n)) = 0$. Hence we obtain $\lim_n d(x_n, y_n) = 0$ from (τ5). ∎

LEMMA 3. *Let X be a metric space with metric d and let p be a τ-distance on X. If a sequence $\{x_n\}$ of X satisfies $\lim_n \sup\{p(x_n, x_m) : m > n\} = 0$, then $\{x_n\}$ is a p-Cauchy sequence. Moreover if a sequence $\{y_n\}$ of X satisfies $\lim_n p(x_n, y_n) = 0$, then $\{y_n\}$ is also a p-Cauchy sequence and $\lim_n d(x_n, y_n) = 0$.*

Proof. Let η be a function from $X \times \mathbb{R}_+$ into $\mathbb{R}_+$ satisfying (τ2)–(τ5). If a sequence $\{x_n\}$ of X satisfies $\lim_n \sup\{p(x_n, x_m) : m > n\} = 0$, then we put $\alpha_n = \sup\{p(x_i, x_j) : j > i \ge n\}$. Note that $\lim_n \alpha_n = 0$. Let $\{x_{f(n)}\}$ be an arbitrary subsequence of $\{x_n\}$. By assumption and (τ2), there exists a subsequence $\{x_{f\circ g(n)}\}$ of $\{x_{f(n)}\}$ such that $\lim_n \eta(x_{f\circ g(n)}, \alpha_{f\circ g(n+1)}) = 0$. Since

$$\limsup_{n\to\infty} \sup_{m \ge n} p\big(x_{f\circ g(n)}, x_{f\circ g(m+1)}\big) \le \lim_{n\to\infty} \alpha_{f\circ g(n)} = 0,$$

we obtain

$$\lim_{n \to \infty} \eta\big(x_{f \circ g(n)}, \alpha_{f \circ g(n)}\big) = \lim_{n \to \infty} \eta\big(x_{f \circ g(n+1)}, \alpha_{f \circ g(n+1)}\big) = 0$$

from (τ4). Since $\{x_{f(n)}\}$ is arbitrary, we have $\lim_n \eta(x_n, \alpha_n) = 0$. Therefore we obtain

$$\limsup_{n \to \infty} \sup_{m \geq n} \eta\big(x_{n-1}, p(x_{n-1}, x_m)\big) \leq \lim_{n \to \infty} \eta\big(x_{n-1}, \alpha_{n-1}\big) = 0.$$

So, $\{x_n\}$ is a p-Cauchy sequence. Moreover if a sequence $\{y_n\}$ of X satisfies $\lim_n p(x_n, y_n) = 0$, then we have

$$\begin{aligned} &\limsup_{n \to \infty} \sup_{m \geq n} p(x_n, y_m) \\ &\quad \leq \limsup_{n \to \infty} \left(\max\left\{ p(x_n, y_n), \sup_{m > n} \big(p(x_n, x_m) + p(x_m, y_m)\big) \right\} \right) \\ &\quad \leq \lim_{n \to \infty} \left(\alpha_n + \sup_{m \geq n} p(x_m, y_m) \right) = 0. \end{aligned}$$

So by Lemma 1, $\{y_n\}$ is also a p-Cauchy sequence and $\lim_n d(x_n, y_n) = 0$. ∎

Remark. In general, a τ-distance p does not necessarily satisfy $p(z, z) = 0$. So Lemma 2 is not a special case of Lemma 3.

4. FIXED POINT THEOREMS

In this section, we prove fixed point theorems. We first prove the following theorem.

THEOREM 1. *Let X be a complete metric space and let T be a mapping from X into itself. Suppose that there exist a τ-distance p on X and $r \in [0, 1)$ such that $p(Tx, T^2x) \leq r \cdot p(x, Tx)$ for all $x \in X$. Assume that either of the following holds*:

(i) *If* $\lim_n \sup\{p(x_n, x_m) : m > n\} = 0$, $\lim_n p(x_n, Tx_n) = 0$, *and* $\lim_n p(x_n, y) = 0$, *then* $Ty = y$;

(ii) *If* $\{x_n\}$ *and* $\{Tx_n\}$ *converge to* y, *then* $Ty = y$;

(iii) *T is continuous.*

Then there exists $x_0 \in X$ such that $Tx_0 = x_0$ and $p(x_0, x_0) = 0$.

Proof. We first prove (ii) implies (i). Suppose that $\lim_n \sup\{p(x_n, x_m) : m > n\} = 0$, $\lim_n p(x_n, Tx_n) = 0$, and $\lim_n p(x_n, y) = 0$. By Lemma 3, we have $\lim_n Tx_n = \lim_n x_n = y$. So we have $Ty = y$ by (ii). This shows (ii) implies (i). Clearly (iii) implies (ii). Finally we shall prove T has a fixed point x_0 with $p(x_0, x_0) = 0$ in the case of (i). Fix $u \in X$ and put $u_n = T^n u$ for all $n \in \mathbb{N}$. Then if $m > n$, we have

$$p(u_n, u_m) \leq \sum_{k=n}^{m-1} p(u_k, u_{k+1}) \leq \sum_{k=n}^{m-1} r^k p(u, u_1) \leq \frac{r^n}{1-r} p(u, u_1)$$

and hence $\lim_n \sup\{p(u_n, u_m) : m > n\} = 0$. By Lemma 3, $\{u_n\}$ is a p-Cauchy sequence and hence it is a Cauchy sequence. Since X is complete, $\{u_n\}$ converges to some point $x_0 \in X$. From (τ3), we have

$$\begin{aligned}
&\limsup_{n\to\infty} \left(p(u_n, Tu_n) + p(u_n, x_0) \right) \\
&\qquad \leq \limsup_{n\to\infty} \Big(p(u_n, u_{n+1}) + \liminf_{m\to\infty} p(u_n, u_m) \Big) \\
&\qquad \leq 2 \lim_{n\to\infty} \sup_{m>n} p(u_n, u_m) = 0.
\end{aligned}$$

Hence $Tx_0 = x_0$ from (i). Further we obtain $p(x_0, x_0) = p(Tx_0, T^2x_0) \leq r \cdot p(x_0, Tx_0) = r \cdot p(x_0, x_0)$ and hence $p(x_0, x_0) = 0$. ∎

The following theorem is a generalization of Theorem 2 in [11], which is the fixed point theorem for contractive mappings with respect to a w-distance.

THEOREM 2. *Let X be a complete metric space and let T be a mapping from X into itself. Suppose T is a contractive mapping with respect to a τ-distance p on X, i.e., there exists $r \in [0,1)$ such that $p(Tx, Ty) \leq r \cdot p(x, y)$ for all $x, y \in X$. Then T has a unique fixed point $x_0 \in X$. Further such x_0 satisfies $p(x_0, x_0) = 0$.*

Proof. Clearly $p(Tx, T^2x) \leq r \cdot p(x, Tx)$ for all $x \in X$. Let us prove (i) in Theorem 1. Suppose $\lim_n \sup\{p(x_n, x_m) : m > n\} = 0$, $\lim_n p(x_n, Tx_n) = 0$, and $\lim_n p(x_n, y) = 0$. From

$$\begin{aligned}
\limsup_{n\to\infty} p(x_n, Ty) &\leq \limsup_{n\to\infty} \left(p(x_n, Tx_n) + p(Tx_n, Ty) \right) \\
&\leq \lim_{n\to\infty} \left(p(x_n, Tx_n) + r \cdot p(x_n, y) \right) = 0,
\end{aligned}$$

we obtain $Ty = y$ by Lemma 3. By Theorem 1, there exists $x_0 \in X$ such that $Tx_0 = x_0$ and $p(x_0, x_0) = 0$. If $y_0 = Ty_0$, then we have $p(x_0, y_0) = p(Tx_0, Ty_0) \leq r \cdot p(x_0, y_0)$ and hence $p(x_0, y_0) = 0$. So, by $p(x_0, x_0) = 0$ and Lemma 2, we obtain $x_0 = y_0$. ∎

5. EXISTENCE THEOREMS

In this section, we prove some existence theorems. We first prove the following proposition. The proof employs the methods in [9].

PROPOSITION 8. *Let X be a complete metric space, let p be a τ-distance on X, and let f be a function from X into $(-\infty, \infty]$ which is proper lower semicontinuous and bounded from below. Define $Mx = \{y \in X : f(y) + p(x, y) \leq f(x)\}$ for all $x \in X$. Then for each $u \in X$ with $Mu \neq \emptyset$, there exists $x_0 \in Mu$ such that $Mx_0 \subset \{x_0\}$. In particular, there exists $y_0 \in X$ such that $My_0 \subset \{y_0\}$.*

Before proving it, we prove the following lemma.

LEMMA 4. *Let X be a metric space, let p be a τ-distance on X, and let f be a function from X into $(-\infty, \infty]$ which is proper lower semicontinuous and bounded from below. Define Mx as in Proposition 8. Let $u \in X$ and $c \in \mathbb{R}_+$ such that $f(u) < \infty$, $Mu \neq \emptyset$, and $c \geq f(u) - \inf f(Mu)$. Then a function q from $X \times X$ into $\mathbb{R}_+$ defined by*

$$q(x, y) = \begin{cases} f(x) - \inf f(Mx), & \text{if } x \in Mu \text{ and } y \in Mx, \\ c + p(x, y), & \text{if } x \notin Mu \text{ or } y \notin Mx \end{cases}$$

is a τ-distance on X.

Proof. Let η be a function from $X \times \mathbb{R}_+$ into $\mathbb{R}_+$ satisfying $(\tau 2)$–$(\tau 5)$. Note that $y \in Mx$ and $z \in My$ imply $z \in Mx$ because

$$f(x) + p(x, z) \leq f(z) + p(x, y) + p(y, z) \leq f(y) + p(x, y) \leq f(x).$$

If $x \in Mu$ and $y \in Mx$, then

$$\begin{aligned} p(x, y) \leq f(x) - f(y) \leq q(x, y) &= f(x) - \inf f(Mx) \\ &\leq f(u) - \inf f(Mu) \leq c. \end{aligned}$$

Therefore $p(x, y) \leq q(x, y) \leq c + p(x, y)$ for all $x, y \in X$. So, to complete the proof, we show $(\tau 1)_q$ and $(\tau 3)_{q,\eta}$. Fix $x, y, z \in X$. In the case of $x \in Mu$, $y \in Mx$, $y \in Mu$, and $z \in My$, we have $z \in Mx$ and hence $q(x, z) = q(x, y) \leq q(x, y) + q(y, z)$. In the other case, we obtain

$$q(x, z) \leq c + p(x, z) \leq c + p(x, y) + p(y, z) \leq q(x, y) + q(y, z).$$

This shows $(\tau 1)_q$. Suppose $\lim_n x_n = x$ and $\lim_n \sup\{\eta(z_n, q(z_n, x_m)) : m \geq n\} = 0$, and fix $w \in X$. Then since $\lim_n \sup\{\eta(z_n, p(z_n, x_m)) : m \geq n\} = 0$, we have $p(w, x) \leq \liminf_n p(w, x_n)$. In the case that $w \in Mu$ and there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \in Mw$ for all $k \in \mathbb{N}$, we have $x \in Mw$ because

$$\begin{aligned} f(x) + p(w,x) &\le \liminf_{n\to\infty} f(x_n) + \liminf_{n\to\infty} p(w,x_n) \\ &\le \liminf_{n\to\infty} \big(f(x_n) + p(w,x_n)\big) \\ &\le \liminf_{k\to\infty} \big(f(x_{n_k}) + p(w,x_{n_k})\big) \le f(w). \end{aligned}$$

Hence

$$q(w,x) = f(w) - \inf f(Mw) = \lim_{k\to\infty} q(w,x_{n_k}) = \liminf_{n\to\infty} q(w,x_n).$$

In the other case, we obtain

$$q(w,x) \le c + p(w,x) \le \liminf_{n\to\infty} \big(c + p(w,x_n)\big) = \liminf_{n\to\infty} q(w,x_n).$$

This shows $(\tau 3)_{q,\eta}$. ∎

Proof of Proposition 8. Fix $u \in X$ with $Mu \ne \varnothing$ and choose $u_1 \in Mu$ with $f(u_1) < \infty$. If $Mu_1 = \varnothing$, the assertion holds. We assume $Mu_1 \ne \varnothing$ and $Mx \cap (X \setminus \{x\}) \ne \varnothing$ for all $x \in Mu_1$. Fix $u_2 \in Mu_1$. From $f(y) \le f(x)$ for all $x \in X$ and $y \in Mx$, we can define a mapping T from X into itself as follows: For each $x \in Mu_1$, Tx satisfies $Tx \in Mx$, $Tx \ne x$ and $f(Tx) \le (f(x) + \inf f(Mx))/2$. For each $x \notin Mu_1$, define $Tx = u_2 \ne x$. We also define a function q from $X \times X$ into $\mathbb{R}_+$ by

$$q(x,y) = \begin{cases} f(x) - \inf f(Mx), & \text{if } x \in Mu_1 \text{ and } y \in Mx, \\ 2f(u_1) - 2\inf f(Mu_1) + 1 + p(x,y), & \text{if } x \notin Mu_1 \text{ or } y \notin Mx. \end{cases}$$

From Lemma 4, q is a τ-distance on X. From the proof of Lemma 4, we know that $y \in Mx$ and $z \in My$ imply $z \in Mx$. Hence we have $Tx \in Mu_1$ and $MTx \subset Mx$ for all $x \in Mu_1$. If $x \in Mu_1$, we obtain

$$\begin{aligned} q(Tx, T^2x) &= f(Tx) - \inf f(MTx) \le f(Tx) - \inf f(Mx) \\ &\le \big(f(x) - \inf f(Mx)\big)/2 = q(x,Tx)/2. \end{aligned}$$

If $x \notin Mu_1$,

$$\begin{aligned} q(Tx, T^2x) &= q(u_2, Tu_2) = f(u_2) - \inf f(Mu_2) \\ &\le f(u_1) - \inf f(Mu_1) \le q(x,u_2)/2 = q(x,Tx)/2. \end{aligned}$$

Let us prove (i) in Theorem 1. Suppose $\lim_n \sup\{q(x_n,x_m) : m > n\} = 0$ and $\lim_n q(x_n,y) = 0$. By the definition of q, we may assume $x_n \in Mu_1$

and $y \in Mx_n$ for all $n \in \mathbb{N}$. Then $y \in Mu_1$ and hence $Ty \in My \subset Mx_n$. We have $\lim_n q(x_n, Ty) = \lim_n q(x_n, y) = 0$ and hence $Ty = y$ by Lemma 3. So, by Theorem 1, T has a fixed point. This is a contradiction. Therefore there exists $x_0 \in Mu_1 \subset Mu$ such that $Mx_0 \subset \{x_0\}$. ∎

Using Proposition 8, we improve Theorem 2 in [5], which is a generalization of Caristi's fixed point theorem [2].

THEOREM 3. *Let X be a complete metric space and let p be a τ-distance on X. Let T be a mapping from X into itself and let f be a function from X into $(-\infty, \infty]$ which is proper lower semicontinuous and bounded from below. Assume $f(Tx) + p(x, Tx) \le f(x)$ for all $x \in X$. Then there exists $x_0 \in X$ such that $Tx_0 = x_0$ and $p(x_0, x_0) = 0$.*

Proof. For each $x \in X$, we define Mx as in Proposition 8. Then there exists $x_0 \in X$ such that $Mx_0 \subset \{x_0\}$ by Proposition 8. From $Tx_0 \in Mx_0$, we obtain $Tx_0 = x_0$. If $f(x_0) = \infty$, then $X = Mx_0 \subset \{x_0\}$. This is a contradiction. Since $f(x_0) < \infty$ and $f(x_0) + p(x_0, x_0) = f(Tx_0) + p(x_0, Tx_0) \le f(x_0)$, we obtain $p(x_0, x_0) = 0$. ∎

As a direct consequence of Theorem 3, we obtain the following.

COROLLARY 1 (Jachymski [4]). *Let X be a complete metric space with metric d and let φ be a nondecreasing and left continuous function from $\mathbb{R}_+$ into itself such that $0 < \varphi(s + t) \le \varphi(s) + \varphi(t)$ for all $s \ge 0$ and $t > 0$. Let T be a mapping from X into itself and let f be a function from X into $(-\infty, \infty]$ which is proper lower semicontinuous and bounded from below. Assume $f(Tx) + \varphi(d(x, Tx)) \le f(x)$ for all $x \in X$. Then there exists $x_0 \in X$ such that $Tx_0 = x_0$.*

Remark. It is not needed that φ is continuous at 0; see [4].

We next improve Theorem 3 in [5], which is a generalization of Ekeland's variational principle [3].

THEOREM 4. *Let X be a complete metric space, let p be a τ-distance on X, and let f be a function from X into $(-\infty, \infty]$ which is proper lower semicontinuous and bounded from below. Then the following* (i) *and* (ii) *hold*:

(i) *For each $u \in X$, there exists $v \in X$ such that $f(v) \le f(u)$ and $f(w) > f(v) - p(v, w)$ for all $w \in X$ with $w \ne v$;*

(ii) *for each $\varepsilon > 0$ and $u \in X$ with $p(u, u) = 0$, there exists $v \in X$ such that $f(v) \le f(u) - \varepsilon p(u, v)$ and $f(w) > f(v) - \varepsilon p(v, w)$ for all $w \in X$ with $w \ne v$.*

Proof. We first prove (i). For each $x \in X$, we define Mx as in Proposition 8. If $Mu = \emptyset$, such u itself satisfies $f(w) > f(u) - p(u, w)$ for all $w \in X$ with $w \ne u$. If $Mu \ne \emptyset$, then there exists $v \in Mu$ such that

$Mv \subset \{v\}$ by Proposition 8. Since $v \in Mu$ implies $f(v) \leq f(u)$, and $Mv \subset \{v\}$ shows that $f(w) > f(v) - p(v,w)$ for all $w \in X$ with $w \neq v$, the assertion holds. Let us prove (ii). Note that εp is a τ-distance by Proposition 7. For each $x \in X$, we define Mx by $Mx = \{y \in X : f(y) + \varepsilon p(x,y) \leq f(x)\}$. Since $p(u,u) = 0$ implies $Mu \neq \emptyset$, there exists $v \in Mu$ such that $Mv \subset \{v\}$ by Proposition 8. Such v satisfies $f(v) \leq f(u) - \varepsilon p(u,v)$ and $f(w) > f(v) - \varepsilon p(v,w)$ for all $w \in X$ with $w \neq v$. This completes the proof. ∎

As direct consequences of Theorem 4, we obtain the following.

COROLLARY 2 (Tataru [14]). *Let X be a closed subset of a Banach space, let $\{T(t) : t \in \mathbb{R}_+\}$ be a strongly continuous semigroup of nonexpansive mappings on X, and let p be Tataru's distance on X. Let f be a function from X into $(-\infty, \infty]$ which is proper lower semicontinuous and bounded from below. Then for each $\varepsilon > 0$ and $u \in X$ there exists $v \in X$ such that $f(v) \leq f(u) - \varepsilon p(u,v)$ and $f(w) > f(v) - \varepsilon p(v,w)$ for all $w \in X$ with $w \neq v$.*

COROLLARY 3 (Zhong [16]). *Let X be a complete metric space with metric d, let h be a nondecreasing function from $\mathbb{R}_+$ into itself such that $\int_0^\infty (1/(1+h(r)))\,dr = \infty$, and let $z_0 \in X$ be fixed. Let f be a funciton from X into $(-\infty, \infty]$ which is proper lower semicontinuous and bounded from below. Then for each $\varepsilon > 0$, $\lambda > 0$, and $u \in X$ with $f(u) \leq \inf\{f(x) : x \in X\} + \varepsilon$, there exists $v \in X$ such that $f(v) \leq f(u)$.*

$$\int_{d(z_0,u)}^{d(z_0,u)+d(u,v)} \frac{dr}{1+h(r)} \leq \lambda$$

and

$$f(w) > f(v) - \frac{\varepsilon}{\lambda} \cdot \frac{d(v,w)}{1+h\big(d(z_0,v)\big)}$$

for all $w \in X$ with $w \neq v$.

Remark. The continuity of h is not needed; see [16].

Proof. Define a τ-distance p on X by

$$p(x,y) = \int_{d(z_0,x)}^{d(z_0,x)+d(x,y)} \frac{dr}{1+h(r)}$$

for all $x, y \in X$. By Theorem 4, there exists $v \in X$ such that $f(v) \leq f(u) - (\varepsilon/\lambda)p(u,v)$ and $f(w) > f(v) - (\varepsilon/\lambda)p(v,w)$ for all $w \in X$ with $w \neq v$.

Therefore $f(v) \le f(u)$ and

$$\begin{aligned}\int_{d(z_0,u)}^{d(z_0,u)+d(u,v)} \frac{dr}{1+h(r)} &= p(u,v) \\ &\le \frac{\lambda}{\varepsilon}(f(u) - f(v)) \\ &\le \frac{\lambda}{\varepsilon}\Big(f(u) - \inf_{x\in X} f(x)\Big) \le \lambda.\end{aligned}$$

We also obtain

$$\begin{aligned} f(w) &> f(v) - \frac{\varepsilon}{\lambda} p(v,w) \\ &= f(v) - \frac{\varepsilon}{\lambda}\int_{d(z_0,v)}^{d(z_0,v)+d(v,w)} \frac{dr}{1+h(r)} \\ &\ge f(v) - \frac{\varepsilon}{\lambda}\cdot\frac{d(v,w)}{1+h\big(d(z_0,v)\big)}\end{aligned}$$

for all $w \in X$ with $w \ne v$. This completes the proof. ∎

Finally we improve Theorem 1 in [5], which is a generalization of Takahashi's theorem [12].

THEOREM 5. *Let X be a complete metric space and let f be a function from X into $(-\infty, \infty]$ which is proper lower semicontinuous and bounded from below. Assume that there exists a τ-distance p on X such that for each $u \in X$ with $f(u) > \inf\{f(x) : x \in X\}$, there exists $v \in X$ with $v \ne u$ and $f(v) + p(u,v) \le f(u)$. Then there exists $x_0 \in X$ such that $f(x_0) = \inf\{f(x) : x \in X\}$.*

Proof. For each $x \in X$, we define Mx as in Proposition 8. Assume that $f(x) > \inf\{f(x) : x \in X\}$ for all $x \in X$. Then by the assumption, $Mx \cap (X \setminus \{x\}) \ne \emptyset$ for all $x \in X$. This contradicts Proposition 8. ∎

As a direct consequence of Theorem 5, we obtain the following, which is a slight generalization of the results of Ume [15], Takahashi [13], and Kim *et al*. [6].

COROLLARY 4. *Let X be a complete metric space with metric d and let f be a function from X into $(-\infty, \infty]$ which is proper lower semicontinuous and bounded from below. Let T be a mapping from X into itself satisfying that $\lim_n x_n = y$ and $\lim_n Tx_n = y$ imply $Ty = y$. Assume that for each $u \in X$ with $f(u) > \inf\{f(x) : x \in X\}$, there exists $v \in X$ with $v \ne u$ and $f(v) + \max\{d(Tu, v), d(Tu, Tv)\} \le f(u)$. Then there exists $x_0 \in X$ such that $f(x_0) = \inf\{f(x) : x \in X\}$.*

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